Electromagnetic

Lecture (9)

مركز الشرق للخدمات العلمية
١١٠٠٩٣٦٤٠٠٦

Chapter (6)

"Laplace Equation"

2nd year

communications

Lecture (9):

Chapter (6)

Laplace Equation

[I] Derivation of Laplace eq^n:

⇛ Gauss law: $\nabla \cdot \bar{D} = \rho_v$

$\bar{D} = \epsilon \bar{E}$ $\quad \therefore \nabla \cdot \bar{E} = \dfrac{\rho_v}{\epsilon}$

$\bar{E} = -\nabla V$ $\quad \therefore \nabla \cdot (\nabla V) = -\dfrac{\rho_v}{\epsilon}$

$\because$ Div (grad) $\equiv$ Laplacian

$\nabla \cdot (\nabla V) \equiv \nabla^2 V$

$$\therefore \nabla^2 V = -\frac{\rho_v}{\epsilon}$$

Laplace assume that $\rho_v = 0$

$$\therefore \boxed{\nabla^2 V = 0} \quad \text{Laplace eq}^n.$$

2] Solution steps :

(1) نحدد نوع إحداثيات المسألة : (Cart, Cyl, Sph)،

(2) نحدد ال V تتغير مع أى parameter من الإحداثي

(3) نحسب معادلة ال V عن طريق التكامل مرتين لمعاد Laplace :

$$\nabla^2 V = 0$$

(4) $\bar{E} = -\nabla V$.

(5) $\bar{D} = \epsilon \bar{E}$.

(6) $\rho_s = |\bar{D}|$.

(7) $Q = \int_s \rho_s \, ds = \underset{\hookrightarrow \text{Const.}}{\rho_s} . S$

(8) $C = \dfrac{Q}{V}$

(9) $W_E = \dfrac{\epsilon}{2} \int_v |E|^2 dv = \dfrac{\epsilon}{2} \underset{\hookrightarrow \text{Const.}}{|E|^2} . \text{Vol}$

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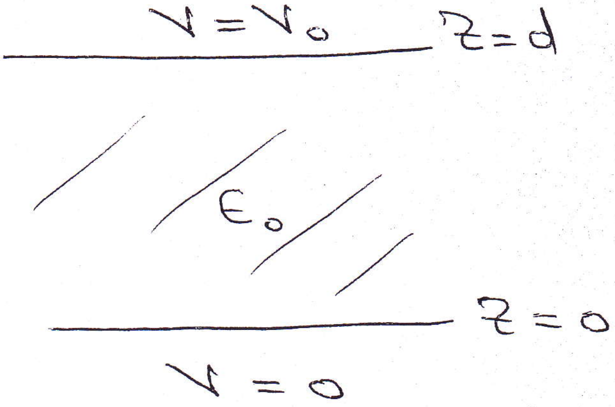
. مسألة Laplace يمكن تمييزها بمعطياتها فلا بد أن تعطى فى المسألة قيم الجهد عند ال Boundaries (B.Cs)

Sheet (6)

[1] area = A



→ using Cart Coo System

→ V varies with z.

→ Laplace eqⁿ : $\nabla^2 V = 0$

$$\therefore \frac{\partial^2 V}{\partial z^2} = 0$$

Int. twice : $\frac{\partial V}{\partial z} = A$

$$V = Az + B$$

← حساب الثوابت باستخدام ال B.Cs :

at $z = 0 \longrightarrow V = 0$

$$\therefore B = 0$$

at $z = d \longrightarrow V = V_0$

$$\therefore A = \frac{V_0}{d}$$

$$\therefore \boxed{V = \frac{V_0}{d} z}$$

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∘ $\bar{E} = -\nabla V = -\left[\frac{\partial V}{\partial z}\hat{z}\right] = -\frac{V_o}{d}\hat{z}$

∘ $\bar{D} = \epsilon_o \bar{E} = -\frac{\epsilon_o V_o}{d}\hat{z}$

∘ $\rho_s = |D| = \frac{\epsilon_o V_o}{d}$

∘ $Q = \rho_s A = \frac{\epsilon_o V_o}{d} A$

$$\therefore C = \frac{Q}{V_o} = \frac{\frac{\epsilon_o V_o}{d} A}{V_o} = \frac{\epsilon_o A}{d}$$

$$\boxed{C = \frac{\epsilon_o A}{d}}$$

∘ $W_E = \frac{\epsilon}{2}\int_v |E|^2 \, dv$

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$\because |E| = \frac{V_o}{d} = \text{const.}$

$\therefore W_E = \frac{\epsilon}{2} \cdot |E|^2 \cdot Vol$

$Vol = A d$ (مساحة القاعدة * الارتفاع)

$$\therefore W_E = \frac{\epsilon_0}{2}\cdot\frac{V_0^2}{d^{\cancel{2}}}\cdot A\cancel{d}$$

$$= \frac{\epsilon_0 A}{2d}V_0^2 \qquad C = \frac{\epsilon_0 A}{d}$$

$$= \frac{1}{2}CV_0^2$$

$$\therefore \boxed{W_E = \frac{CV_0^2}{2}}$$

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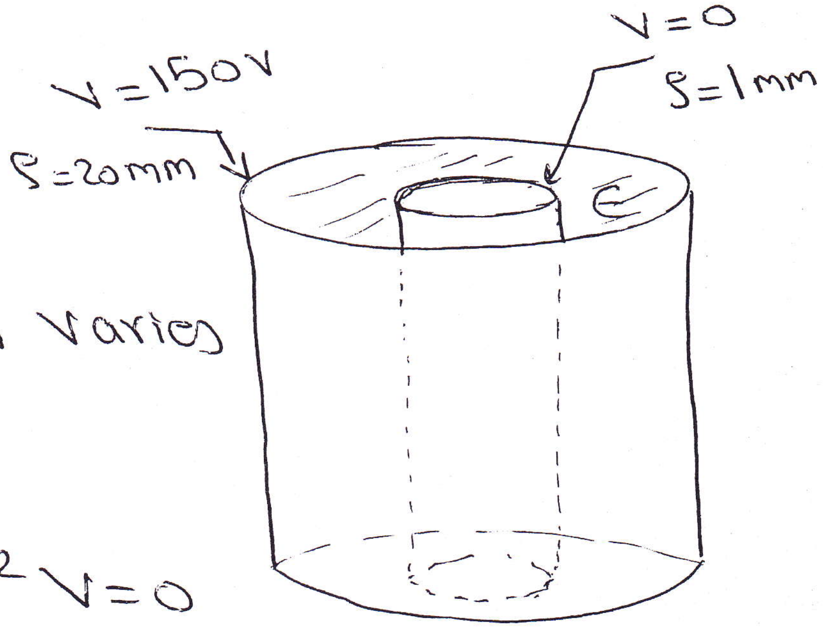
2)



- Using Cyl. Coo. (V varies with ρ).
- Laplace eq^n : $\nabla^2 V = 0$

$$\therefore \frac{1}{\cancel{\rho}}\frac{d}{d\rho}\left(\rho\frac{dV}{d\rho}\right) = 0$$

$$\therefore \rho\frac{dV}{d\rho} = A \Rightarrow \frac{dV}{d\rho} = \frac{A}{\rho}$$

$\therefore V = A \ln \rho + B$ (تكامل مرتين)

حساب الثوابت:

at $\rho = 1\,mm$ $V = 0$

$\therefore 0 = -6.91 A + B$

$\therefore B = 6.91 A \longrightarrow (1)$

at $\rho = 20\,mm$ $V = 150\,V$

$\therefore 150 = -3.91 A + B \longrightarrow (2)$

from (1) in (2):

$\therefore 150 = -3.91 A + 6.91 A$

$\therefore A = 50$

$B = 345.5$

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$\therefore \boxed{V = 50 \ln \rho + 345.5}$

~~$E = -\nabla V = -\left[\frac{1}{\rho}\frac{\partial V}{\partial \rho}\hat{a}_\rho\right]$~~

$$\bullet\ \bar{E} = -\nabla V = -\left[\frac{\partial V}{\partial \rho}\hat{\rho}\right]$$

$$= -\left[\frac{50}{\rho}\hat{\rho}\right]$$

$$\bullet\ \bar{D} = \epsilon\bar{E} = -\frac{50\,\epsilon}{\rho}\hat{\rho}$$

$$\bullet\ \rho_s = |D| = \frac{50\,\epsilon}{\rho}$$

$\rho = 20\text{mm}$ → Sur. $(V = 150\,V)$

$$\therefore \rho_s = \frac{50\,\epsilon}{0.02}$$

$\rho = 1\text{mm}$ → Sur. $(V = 0\,V)$

$$\rho_s = \frac{50\,\epsilon}{0.001}$$

$\bullet\ Q = \rho_s \cdot S$ (upper Cylinder)

$$\therefore Q = \frac{50\,\epsilon}{\cancel{0.02}} * 2\pi(\cancel{0.02})\,l$$

$$= 100\,\pi\,\epsilon\,l$$

$$\therefore C = \frac{Q}{V_0} = \frac{100\,\pi\,\epsilon\,l}{150} = \frac{2\pi}{3}\epsilon\,l.$$

$\therefore \bar{E} = -\frac{50}{\rho}\hat{\rho}$ and $C = \frac{2\pi\epsilon}{3}\,l$

$\boxed{3}$ **For fig. 1:**

- Using Cyl. Coo. (V varies with ϕ)

- Laplace eqn: $\nabla^2 V = 0$

$$\therefore \frac{1}{\cancel{\rho}} \frac{d^2 V}{d\phi^2} = 0$$

$$\therefore V = A\phi + B$$

- at $\phi = 0$: $V = 0$

$$\therefore B = 0$$

at $\phi = \alpha$: $V = 100$ V

$$\therefore 100 = A\,\alpha$$

$$\therefore A = \frac{100}{\alpha}$$

$$\therefore \boxed{V = \frac{100}{\alpha}\phi}$$

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- $\bar{E} = -\nabla V = -\left[\frac{1}{\rho} \frac{\partial V}{\partial \phi} \hat{\phi}\right]$

$$\therefore \bar{E} = -\frac{1}{\rho}\cdot\frac{100}{\alpha}\hat{\phi}$$

$$\boxed{\bar{E} = -\frac{100}{\rho\alpha}\hat{\phi}}$$

- $\bar{D} = \epsilon\bar{E} = -\dfrac{100\epsilon}{\rho\alpha}\hat{\phi}$
- $\rho_s = |D| = \dfrac{100\epsilon}{\rho\alpha}$
- $Q = \rho_s\cdot S = \dfrac{100\epsilon}{\rho\alpha}\cdot S$
- $C = \dfrac{Q}{V_0} = \dfrac{\cancel{100}\epsilon\cdot S}{\rho\alpha\cdot\cancel{100}} = \dfrac{\epsilon S}{\rho\alpha}$

$$\boxed{C = \frac{\epsilon S}{\rho\alpha}}$$

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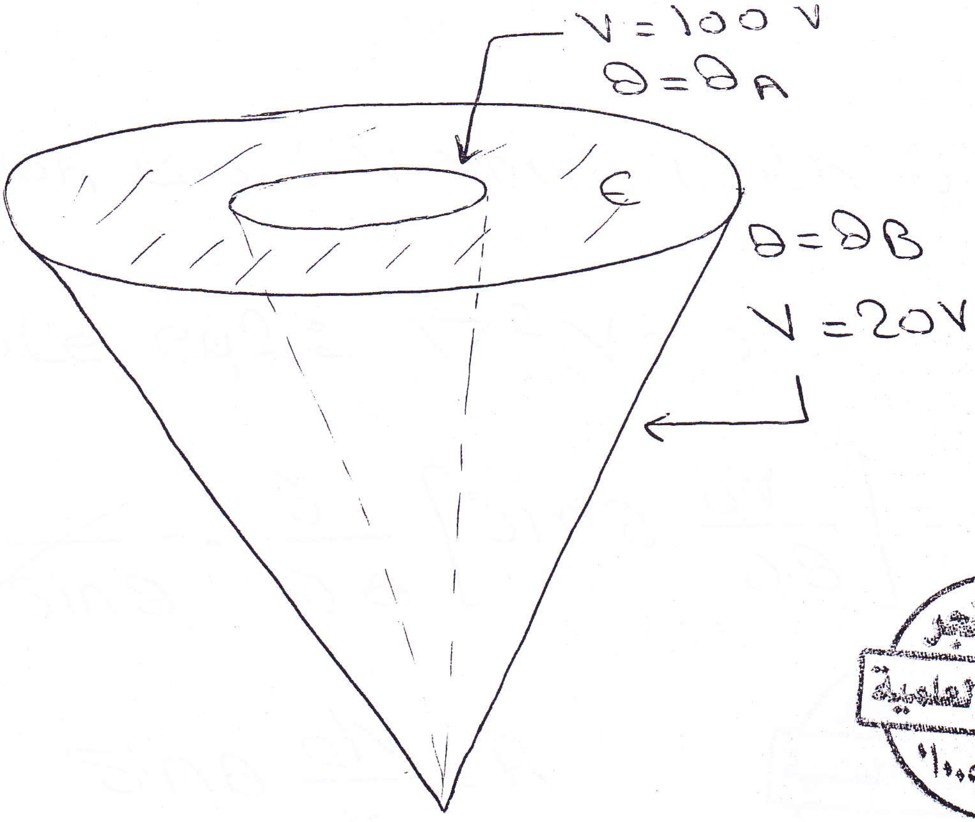
$\boxed{4}$



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Cone A : Point A(1, 0, 2).

Cone B : Poinb B(0, 3, 2).

Find $\bar{E}$, V at point P(1, 1, 1)

Solution

$$\because \theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z}$$ علاقة ال Sph.
بال Cart.

$$\therefore \theta_A = \tan^{-1} \frac{\sqrt{1 + 0}}{2} = 26.56^\circ$$

$$\theta_B = \tan^{-1} \frac{\sqrt{0 + 9}}{2} = 56.31^\circ$$

$$\theta_P = \tan^{-1} \frac{\sqrt{1 + 1}}{} = 54.73^\circ$$

• Using sph Coo. (V varies with θ)

$\therefore$ Laplace eqn: $\nabla^2 V = 0$

$$\therefore \cancel{\frac{1}{r^2 \sin\theta}} \cdot \frac{\partial}{\partial\theta}\left[\sin\theta \frac{\partial V}{\partial\theta}\right] = 0$$

$$\therefore \sin\theta \frac{dV}{d\theta} = A$$

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$$\therefore \frac{dV}{d\theta} = \frac{A}{\sin\theta}$$

تكامل محفوظ

$$\therefore V = A \ln\left\{\tan\left(\frac{\theta}{2}\right)\right\} + B$$

• حساب الثوابت :

at $\theta = \theta_A = 26.56^\circ$: $V = 100\text{ V}$

$$\therefore 100 = A \ln\left\{\tan\left(\frac{\theta_A}{2}\right)\right\} + B$$

$$100 = -1.44 A + B \longrightarrow (1)$$

at $\theta = \theta_B = 56.31^\circ$: $V = 20\text{ V}$

$$\therefore 20 = -0.63 A + B \longrightarrow (2)$$

بحل المعادلتين (١) و (٢) :

$A = -98.76$

$B = -42.22$

لا نعوض بالثوابت في معادلة الجهد إلا بعد انتهاء المسألة لتسهيل الحسابات.

$$V = A \ln\left[\tan\left(\frac{\theta}{2}\right)\right] + B$$

$$\therefore \bar{E} = -\nabla V = -\frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta}$$

$$= -\frac{1}{r}\cdot\frac{A}{\sin\theta}\hat{\theta}$$

$$\therefore V = -98.76 \ln\left[\tan\left(\frac{\theta}{2}\right)\right] - 42.22$$

$$\bar{E} = \frac{98.76}{r\sin\theta}\hat{\theta}$$

for point $P(1,1,1)$

$\theta_p = 54.73°$

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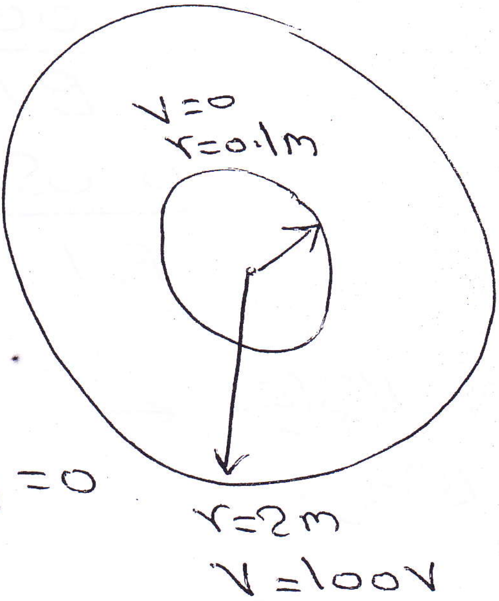
$r_p = \sqrt{x^2 + y^2 + z^2} = \sqrt{3}$

$\therefore V_p = 22.82 \text{ V}$

$$\bar{E}_p = 69.84\, \hat{\theta} \text{ (V/m)}$$

$\boxed{5}$



- Sph Coo. (r).

- $\nabla^2 V = 0$

$$\therefore \frac{1}{\cancel{r^2 \sin\theta}} \cdot \frac{d}{dr}\left(r^2 \cancel{\sin\theta} \frac{dV}{dr}\right) = 0$$

$$\therefore r^2 \frac{dV}{dr} = A$$

$$\frac{dV}{dr} = \frac{A}{r^2}$$

$$\therefore V = -\frac{A}{r} + B$$

- at $r = 0.1$ m. $V = 0$

$$\therefore 0 = -\frac{A}{0.1} + B \longrightarrow (1)$$

$$B = 10A$$

at $r = 2\ m$ $\quad V = 100\ V$

$$\therefore 100 = -\frac{A}{2} + B$$

from (1) : $100 = -\frac{A}{2} + 10A$

$$\therefore A = \frac{200}{19}$$

$$B = \frac{2000}{19}$$

$$\therefore V = -\frac{A}{r} + B \longrightarrow$$ نعوض بالثوابت في الأصل.

$$\circ\ \bar{E} = -\nabla V = -\frac{dV}{dr}\hat{r} = -\frac{A}{r^2}\hat{r}$$

$$\bar{D} = \epsilon\bar{E} = -\frac{\epsilon A}{r^2}\hat{r}$$

$$\rho_s = |D| = \frac{\epsilon A}{r^2}$$

inner sph $\quad \frac{\epsilon A}{(0.1)^2}$

outer sph. $\quad \frac{\epsilon A}{(2)^2}$

take the outer : $\rho_s = \frac{\epsilon A}{4}$

$$\therefore\ Q = \rho_s \cdot S = \frac{\epsilon A}{\cancel{4}} \cdot 4\pi \cancel{(2)^2}$$

$$= 4\pi \epsilon A$$

$$\therefore\ C = \frac{Q}{V_0} = \frac{4\pi\epsilon A}{100} = \frac{\pi \epsilon A}{25}$$

$$\therefore\ C = \frac{\pi\epsilon(200/19)}{25}$$

$$\boxed{C = 1.323\,\epsilon}$$

$$\cdot\ W_E = \frac{\epsilon}{2}\int_v |E|^2\, dv$$

$$\because\ |E| = \frac{A}{r^2}$$

$$\therefore\ W_E = \frac{\epsilon}{2}\int_0^{2\pi}\int_0^{\pi}\int_{0.1}^{2} \frac{A^2}{r^4}\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \frac{\epsilon}{2} A^2 \cdot \frac{-1}{r}\Big|_{0.1}^{2} \cdot \cos\theta\Big|_{\pi}^{0} \cdot 2\pi$$

$$= \checkmark$$

Mansoura University
Faculty of Engineering
Communication Engineering Department

Second Year Communication
Electromagnetic Fields
Sheet (7)

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[1] An air filled parallel plate capacitor is made of two perfectly conducting plates each of area A. The plate separation is d. The voltage between the plates is V_0. Find the capacitance and the energy stored inside the capacitor.

[2] Find the capacitance and electric field intensity for the region between two concentric right circular cylinders, where $V = 0$ at $\rho = 1$ mm and $V = 150$ V at $\rho = 20$ mm.

[3] In cylindrical coordinates two φ = constant planes are insulated along the z axis, as shown in Fig.1, Find the expression for the capacitance and **E** between the planes, assuming a potential of 100 V for $\varphi = \alpha$ and a zero reference at $\varphi = 0$.

[4] Two coaxial cones have their vertices at the origin and the z axis as their axis. Cone A has the point A (1,0,2) on its surface, while cone B has the point B (0,3,2) on its surface. If $V_A = 100$ V and $V_B = 20$ V, find V and **E** at the point P (1, 1, 1).

[5] In spherical coordinates, $V = 0$ for $r = 0.1$ m and $V = 100$ V for $r = 2$ m. Find the capacitance and the stored energy.

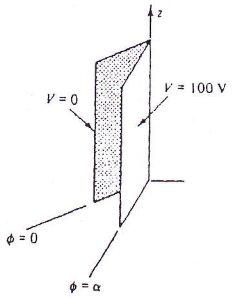

Fig.1